

A compactness theorem for Fueter sections

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Abstract

We prove that a sequence of Fueter sections of a bundle of compact hyperkähler manifolds \mathfrak{X} over a 3-manifold M with bounded energy converges (after passing to a subsequence) outside a 1-dimensional closed rectifiable subset $S \subset M$. The non-compactness along S has two sources: (1) Bubbling-off of holomorphic spheres in the fibres of \mathfrak{X} transverse to a subset $\Gamma \subset S$, whose tangent directions satisfy strong rigidity properties. (2) The formation of non-removable singularities in a set of \mathcal{H}^1 -measure zero. Our analysis is based on the ideas and techniques that Lin developed for harmonic maps [Lin99]. These methods also apply to Fueter sections on 4-dimensional manifolds; we discuss the corresponding compactness theorem in an appendix. We hope that the work in this paper will provide a first step towards extending the hyperkähler Floer theory developed by Hohloch–Noetzel–Salamon [HNS09, Sal13] to general target spaces. Moreover, we expect that this work will find applications in gauge theory in higher dimensions.

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MSC2010: 58E20; 53C26, 53C43

1 Introduction

Let M be an orientable Riemannian 3-manifold, let $\mathfrak{X} \xrightarrow{\pi} M$ be a bundle of hyperkähler manifolds together with a fixed isometric identification $I: STM \rightarrow \mathfrak{H}(\mathfrak{X})$ of the unit tangent bundle in M and the bundle of hyperkähler spheres¹ of the fibres of \mathfrak{X} and fix a connection on \mathfrak{X} .

¹Given a hyperkähler manifold (X, g, I_1, I_2, I_3) , for each $\xi = (\xi_1, \xi_2, \xi_3) \in S^2 \subset \mathbf{R}^3$, $I_\xi := \sum_{i=1}^3 \xi_i I_i$ is a complex structure. The set $\mathfrak{H}(X) := \{I_\xi : \xi \in S^2\}$ is called the *hyperkähler sphere* of X .

Definition 1.1. A section $u \in \Gamma(\mathfrak{X})$ is called a *Fueter section* if

$$(1.2) \quad \mathfrak{F}u := \sum_{i=1}^3 I(v_i) \nabla_{v_i} u = 0 \in \Gamma(u^* V\mathfrak{X})$$

for some local orthonormal frame (v_1, v_2, v_3) .² Here $\nabla u \in \Omega^1(M, u^* V\mathfrak{X})$ is the covariant derivative of u , a 1-form taking values in the pull-back of the vertical tangent bundle $V\mathfrak{X} := \ker(d\pi: T\mathfrak{X} \rightarrow TM)$. The operator \mathfrak{F} is called the *Fueter operator*.

The Fueter operator is a non-linear generalisation of the Dirac operator, see Taubes [Tau99] and Haydys [Hay14, Section 3].

Remark 1.3. A construction similar to (1.2) also exists in dimension four. Since it is more involved, we relegate its discussion to [Appendix B](#).

Example 1.4. Choose a spin structure \mathfrak{s} on M . If $\mathfrak{X} = \mathcal{S}$, I is the Clifford multiplication and ∇ denotes the induced spin connection, then the Fueter operator is simply the Dirac operator associated with \mathfrak{s} .

Example 1.5. Let (X, g, I_1, I_2, I_3) be a hyperkähler manifold and (v_1, v_2, v_3) a orthonormal frame of M . A map $u: M \rightarrow X$ satisfying

$$(1.6) \quad \mathfrak{F}u = \sum_{i=1}^3 I_i du(v_i) = 0$$

is called a Fueter map. In a local trivialisation the Fueter equation for sections of \mathfrak{X} , takes the form (1.6) up to allowing for the I_i to depend on $x \in M$ and admitting a lower order perturbation (coming from the connection 1-form).

One of the main motivations for studying Fueter sections is the work of Hohloch–Noetzel–Salamon [HNS09], who introduced a functional whose critical points are precisely the solution of (1.6) and developed the corresponding Floer theory in the case when the target X is compact and flat, and the frame on M is divergence free and regular³, see also Salamon [Sal13]. The requirement that X be flat is very severe and one would like to remove it. It has been conjectured that the putative hyperkähler Floer theory should be very rich and interesting, especially in the case when X is a $K3$ surface.

²Of course, \mathfrak{F} does not depend on the choice of (v_1, v_2, v_3) .

³Every 3-manifold admits a divergence free frame by Gromov’s h-principle [Sal13, Theorem A.1]. A frame is regular if there are no non-constant Fueter maps $M \rightarrow \mathbf{H}$ with respect to this frame; this is a generic condition.

A further source of motivation is gauge theory on G_2 - and $\text{Spin}(7)$ -manifolds. Here, Fueter sections of bundles of moduli spaces of ASD instantons naturally appear in relation with codimension four bubbling phenomena for G_2 - and $\text{Spin}(7)$ -instantons; see Donaldson–Segal [DS11] and the author [Wal12, Wal14] for further details.

Remark 1.7. Sonja Hohloch brought to the author’s attention a cryptic remark in Kontsevich–Soibelman [KS08, Section 1.5 Question 3], which indicates that their invariants of 3D Calabi–Yau categories with stability structure can be interpreted as “quaternionic Gromov–Witten invariants” of certain hyperkähler manifold \mathcal{M} , which means as a count of Fueter maps from some 4-manifold to \mathcal{M} .

A major issue when dealing with Fueter sections is the potential failure of compactness. This is demonstrated by the following example due to Hohloch–Noetzel–Salamon.

Example 1.8. Consider a $K3$ surface X with a hyperkähler structure such that (X, I_1) admits a non-trivial holomorphic sphere $\mathfrak{z}: S^2 \rightarrow X$ and take $M = \text{SU}(2)$, the unit-sphere in the quaternions \mathbf{H} , with a left-invariant frame (v_1, v_2, v_3) which at $\text{id} \in \text{SU}(2)$ it is given by (i, j, k) . Let $\pi: S^3 \rightarrow S^2$ denote the Hopf fibration whose fibres are the orbits of v_1 . It is easy to check that $u = \mathfrak{z} \circ \pi: S^3 \rightarrow X$ is a Fueter map. For $\lambda > 0$ define a conformal map $s_\lambda: S^2 \rightarrow S^2$ by $s_\lambda(x) = \lambda x$ for $x \in \mathbf{R}^2 \subset S^2$ and $s_\lambda(\infty) = \infty$. Now, the family of Fueter maps $u_\lambda := \mathfrak{z} \circ s_\lambda \circ \pi$ blows up along the Hopf circle $\pi^{-1}(\infty)$ as $\lambda \downarrow 0$ and converges to the constant map on the complement of the Hopf circle.

The following is the main result of this article.

Theorem 1.9. *Suppose \mathfrak{X} is compact. Let (u_i) be a sequence of solutions of the (perturbed) Fueter equation*

$$(1.10) \quad \mathfrak{F}u_i = \mathfrak{p} \circ u_i$$

with $\mathfrak{p} \in \Gamma(\mathfrak{X}, V\mathfrak{X})^4$ and

$$(1.11) \quad \mathcal{E}(u_i) := \int_M |\nabla u_i|^2 \leq c_{\mathcal{E}}$$

for some constant $c_{\mathcal{E}} > 0$. Then (after passing to a subsequence) the following holds:

- There exists a closed subset S with $\mathcal{H}^1(S) < \infty$ and a Fueter section $u \in \Gamma(M \setminus S, \mathfrak{X})$ such that $u_i|_{M \setminus S}$ converges to u in C_{loc}^∞ .

⁴This sort of deformation of (1.2) is important for applications; e.g., Hohloch–Noetzel–Salamon perturb (1.2) using a Hamiltonian function to achieve transversality.

- There exist a constant $\varepsilon_0 > 0$ and an upper semi-continuous function $\Theta : S \rightarrow [\varepsilon_0, \infty)$ such that the sequence of measures $\mu_i := |\nabla u_i|^2 \mathcal{H}^3$ converges weakly to $\mu = |\nabla u|^2 \mathcal{H}^3 + \Theta \mathcal{H}^1 \llcorner S$.
- S decomposes as

$$S = \Gamma \cup \text{sing}(u)$$

with

$$\Gamma := \text{supp}(\Theta \mathcal{H}^1 \llcorner S) \quad \text{and} \quad \text{sing}(u) := \left\{ x \in M : \limsup_{r \downarrow 0} \frac{1}{r} \int_{B_r(x)} |\nabla u|^2 > 0 \right\}.$$

Γ is \mathcal{H}^1 -rectifiable, and $\text{sing}(u)$ is closed and $\mathcal{H}^1(\text{sing}(u)) = 0$.

- For each smooth point⁵ $x \in \Gamma$, there exists a non-trivial holomorphic sphere $\mathfrak{z}_x : S^2 \rightarrow (\mathfrak{X}_x := \pi^{-1}(x), I(v))$ with v a unit tangent vector in $T_x \Gamma$. Moreover,

$$\Theta(x) \geq \mathcal{E}(\mathfrak{z}_x) := \int_{S^2} |\mathrm{d}\mathfrak{z}_x|^2.$$

- If \mathfrak{X} is a bundle of simple hyperkähler manifolds with $b_2 \geq 6$, then there is a subbundle $\mathfrak{d} \subset \mathbf{PTM}$, depending only on $\sup \Theta$, whose fibres are finite sets such that $T_x \Gamma \in \mathfrak{d}$ for all smooth points $x \in \Gamma$.

Remark 1.12. The analysis of (1.2) is similar to Lin's work on the compactness problem for harmonic maps [Lin99]. We follow his strategy quite closely; however, there are a number of simplifications in our case, many of the arguments have to be approached from a different angle and our result is stronger.

Remark 1.13. In the situation of Example 1.5 if X is flat and (v_1, v_2, v_3) is regular, then the uniform energy bound (1.11) is automatically satisfied; see Salamon [Sal13, Lemma 3.2 and Remark 3.5].

Remark 1.14. If I is parallel (which is very rarely the case, but holds, e.g., in the situation of Example 1.5 if $M = T^3$ equipped with a flat metric and the v_i are parallel), then there are topological energy bounds; see Remark 2.10. In this case Fueter sections are stationary harmonic sections and one can derive most of Theorem 1.9 from [Lin99]; cf. Li–Tian [LT98, Section 4] and Chen–Li [CL00], who study triholomorphic/quaternionic maps between hyperkähler manifolds.

⁵We call a point $x \in \Gamma$ smooth if the tangent space $T_x \Gamma$ exists and $x \notin \text{sing}(u)$. Since Γ is rectifiable, $T_x \Gamma$ exists almost everywhere.

Remark 1.15. In the situation of [Example 1.5](#) if X is flat, then $S = \emptyset$; see Hohloch–Noetzel–Salamon [[HNS09](#), Section 3] and [Remark 3.5](#). This does not immediately follow from [Theorem 1.9](#); however, since $\pi_2(T^n) = 0$, flat hyperkähler manifolds admit no non-trivial holomorphic spheres and we can rule out bubbling a priori, i.e., $\Gamma = \emptyset$.

Remark 1.16. By Bogomolov’s decomposition theorem (after passing to a finite cover) any hyperkähler manifold is a product a flat torus and a simple hyperkähler manifold. Hohloch–Noetzel–Salamon’s compactness result says that nothing interesting happens in the torus-factors. Thus the assumption of \mathfrak{X} being a bundle of simple hyperkähler manifolds is not restrictive. The requirement $b_2 \geq 6$ is an artefact of a result of Amerik–Verbitsky we use in [Section 8](#).

As stated, [Theorem 1.9](#) is very likely far from optimal. Here are some conjectural improvements:

- We believe that the limiting section $u \in \Gamma(M \setminus S, \mathfrak{X})$ extends to $M \setminus \text{sing}(u)$ and, moreover, that $\text{sing}(u)$ is finite (possibly countable).
- The holomorphic sphere \mathfrak{z}_x can be replaced by a bubble-tree, cf. Parker–Wolfson [[PW93](#)], such that the energy of the entire bubble tree equals $\Theta(x)$; however, we currently cannot prove that there is no energy stuck on neck regions and that the bubbles have to connect. See Lin–Rivière [[LR02](#)] for progress on a related problem for harmonic maps into spheres.
- We believe that Γ enjoys much better regularity than just being \mathcal{H}^1 –rectifiable. It seems reasonable to expect that Γ is a graph (possibly with countably many vertices) embedded in M and Θ is constant along the edges of Γ ; moreover, we expect that the vertices (Γ, Θ) are balanced. We present some evidence for this in [Section 11](#).

Remark 1.17. In the situation of [Remark 1.14](#), Bethuel’s removable singularities theorem for stationary harmonic maps [[Bet93](#), Theorem I.4] shows that u extends to $M \setminus \text{sing}(u)$ and a result of Allard–Almgren [[AA76](#)] affirms the conjecture in the third bullet.

It is an interesting and important question to ask: what happens for a generic choice of $I: STM \rightarrow \mathfrak{H}(\mathfrak{X})$ and perturbation \mathfrak{p} ? One would hope (perhaps too optimistically) that generically the situation is much better and possibly good enough to count solutions of [\(1.10\)](#) and thus define the Euler characteristic of the conjectural hyperkähler Floer theory.

Assumptions and conventions Throughout the rest of the article we assume the hypotheses of [Theorem 1.9](#). We use c to denote a generic constant. We fix a constant $r_0 > 0$ which is much smaller than the injectivity radius of M and we take all radii to be at most r_0 . $\{\cdot, \dots, \cdot\}$ denotes a generic (multi-)linear expression which is bounded by c .

2 Mononicity formula

The foundation of the analysis of [\(1.2\)](#) is the monotonicity formula which asserts that the renormalised energy

$$\frac{1}{r} \int_{B_r(x)} |\nabla u|^2.$$

is almost monotone in $r > 0$:

Proposition 2.1. *If $u \in \Gamma(M, \mathfrak{X})$ satisfies [\(1.10\)](#), then for all $x \in M$ and $0 < s < r$*

$$\frac{e^{cr}}{r} \int_{B_r(x)} |\nabla u|^2 - \frac{e^{cs}}{s} \int_{B_s(x)} |\nabla u|^2 \geq \int_{B_r(x) \setminus B_s(r)} \frac{1}{\rho} |\nabla_r u|^2 - cr^2.$$

Here $\rho := d(x, \cdot)$.

It is instructive to first prove the following which contains the essence of [Proposition 2.1](#).

Proposition 2.2. *If $u: \mathbf{R}^3 \rightarrow X$ is a Fueter map with $v_i = \frac{\partial}{\partial x^i}$, then for all $x \in M$ and $0 < s < r$*

$$(2.3) \quad \frac{1}{r} \int_{B_r(x)} |du|^2 - \frac{1}{s} \int_{B_s(x)} |du|^2 = 2 \int_{B_r(x) \setminus B_s(r)} \frac{1}{\rho} |\partial_r u|^2.$$

Proof. The derivative of

$$f(\rho) := \frac{1}{\rho} \int_{B_\rho(x)} |du|^2$$

is

$$f'(\rho) = -\frac{1}{\rho^2} \int_{B_\rho(x)} |du|^2 + \frac{1}{\rho} \int_{\partial B_\rho(x)} |du|^2.$$

By a direct computation

$$(2.4) \quad |du|^2 \text{ vol} = |\mathfrak{F}u|^2 \text{ vol} + 2 \sum_{i=1}^3 dx^i \wedge u^* \omega_i,$$

see [HNS09, Lemma 2.2]. Here $\omega_i = g(I_i \cdot, \cdot)$ denotes the Kähler form on X associated with I_i . Hence,

$$(2.5) \quad \begin{aligned} \int_{B_\rho(x)} |du|^2 &= 2 \int_{B_\rho(x)} \sum_{i=1}^3 dx^i \wedge u^* \omega_i = 2 \int_{B_\rho(x)} \sum_{i=1}^3 d(x^i u^* \omega_i) \\ &= 2\rho \int_{\partial B_\rho(x)} u^* \omega_{\partial_r} \end{aligned}$$

with $\partial_r = \sum_{i=1}^3 \frac{x^i}{|x|} \frac{\partial}{\partial x^i}$ denoting the radial vector field. On $\partial B_\rho(x)$, we can take the local orthonormal frame (v_1, v_2, v_3) to be of the form $(\partial_r, \partial_1, \partial_2)$ with (∂_1, ∂_2) a local positive orthonormal frame for $\partial B_\rho(x)$. Now, minus twice the integrand in the last term is

$$(2.6) \quad \begin{aligned} -2\langle I(\partial_r)\partial_1 u, \partial_2 u \rangle &= 2\langle I_1 \partial_1 u, I_2 \partial_2 u \rangle \\ &= |I_1 \partial_1 u + I_2 \partial_2 u|^2 - |I_1 \partial_1 u|^2 - |I_2 \partial_2 u|^2 \\ &= 2|\partial_r u|^2 - |du|^2. \end{aligned}$$

Putting everything together yields

$$f'(\rho) = 2\rho^{-1} \int_{\partial B_\rho} |\partial_r u|^2.$$

Upon integration this yields (2.3). \square

Proof of Proposition 2.1. The map I yields section of $\pi^* TM \otimes \Lambda^2 V\mathfrak{X}$ which, using the connection on \mathfrak{X} , can be viewed as a 3-form $\Lambda \in \Omega^3(\mathfrak{X})$. For sections of \mathfrak{X} the identity (2.4) is replaced by

$$(2.7) \quad |\nabla u|^2 \text{ vol} = |\mathfrak{F}u|^2 \text{ vol} + 2u^* \Lambda.$$

If we define $f(\rho)$ as before, then using (2.7) its derivative can be written as

$$f'(\rho) = -\rho^{-2} \int_{B_\rho(x)} |\mathfrak{p} \circ u|^2 - 2\rho^{-2} \int_{B_\rho(x)} u^* \Lambda + \rho^{-1} \int_{\partial B_\rho(x)} |\nabla u|^2.$$

Let ∂_r denote the radial vector field emanating from x and set $\Omega := i(\underline{v})\Lambda$ with $\underline{v} := \pi^*(r\partial_r)$. We can write Λ as

$$\Lambda = d\Omega + \mathfrak{e}$$

where \mathfrak{e} is the sum of a form of type $(1, 2)$ and a form of type $(2, 1)$ satisfying

$$(2.8) \quad |\mathfrak{e}| = O(\delta \underline{r}) \quad \text{with} \quad \delta := |\nabla I| + |F_{\mathfrak{X}}| + |R|.$$

Here we use the bi-degree decomposition of $\Omega^*(\mathfrak{X})$ arising from $T\mathfrak{X} = \pi^*TM \oplus V\mathfrak{X}$, $\underline{r} := d(x, \pi(\cdot))$, $F_{\mathfrak{X}}$ is the curvature of the connection on \mathfrak{X} and R is the Riemannian curvature of M . Hence,

$$(2.9) \quad \begin{aligned} -2 \int_{B_\rho(x)} u^* \Lambda &= -2 \int_{\partial B_\rho(x)} u^* \Omega + O(\rho^2)f(\rho) + O(\rho^4) \\ &= -2 \int_{\partial B_\rho(x)} i(\partial_r)u^* \Lambda + O(\rho^2)f(\rho) + O(\rho^4). \end{aligned}$$

Arguing as before,

$$-2 \int_{\partial B_\rho(x)} i(\partial_r)u^* \Lambda = \rho \int_{\partial B_\rho(x)} |I_{\partial_r} \nabla_r u - \mathfrak{p} \circ u| + |\nabla_r u|^2 - |\nabla u|^2.$$

Putting everything together one obtains

$$f'(\rho) \geq \rho^{-1} \int_{B_\rho(x)} |\nabla_r u|^2 - cf(\rho) - c\rho.$$

This integrates to prove the assertion. \square

Remark 2.10. If Λ is closed (which is rarely the case), then

$$\mathcal{E}(u) = \int_M |\nabla u|^2 = \langle [M], [u^* \Lambda] \rangle + \int_M |\mathfrak{F}u|^2.$$

Since the first term on the right-hand side only depends on the homotopy class of u , this yields a priori energy bounds for Fueter sections.

Corollary 2.11. *In the situation of [Proposition 2.1](#),*

$$\frac{1}{s} \int_{B_s(x)} |\nabla u|^2 \lesssim \frac{1}{r} \int_{B_r(x)} |\nabla u|^2$$

and if $B_s(y) \subset B_{r/2}(x)$, then

$$\frac{1}{s} \int_{B_s(x)} |\nabla u|^2 \lesssim \frac{1}{r} \int_{B_r(x)} |\nabla u|^2.$$

3 ε -regularity

The following is the key result for proving [Theorem 1.9](#). It allows to obtain local L^∞ -bounds on ∇u provided the renormalised energy is not too large.

Proposition 3.1. *There is a constant $\varepsilon_0 > 0$ such that if $u \in \Gamma(M, \mathfrak{X})$ satisfies (1.10) and*

$$\varepsilon := \frac{1}{r} \int_{B_r(x)} |\nabla u|^2 \leq \varepsilon_0,$$

then

$$(3.2) \quad \sup_{y \in B_{r/4}(x)} |\nabla u|^2(y) \lesssim r^{-2} \varepsilon + 1.$$

Remark 3.3. Given (3.2), higher derivative bounds over slightly smaller balls can be obtained using interior elliptic estimates.

Proposition 3.1 follows from the following differential inequality and Corollary 2.11 using the Heinz trick; see Appendix A.

Proposition 3.4. *If $u \in \Gamma(M, \mathfrak{X})$ satisfies (1.10), then*

$$\Delta |\nabla u|^2 \lesssim |\nabla u|^4 + 1.$$

Proof. This is proved in [HNS09, Lemma 3.3 and Remark 3.4]. We recall the proof which is a simple direct computation. Denote by $\bar{\nabla}$ the induced connection on $u^*V\mathfrak{X}$ and define $F: \Omega^0(M, u^*V\mathfrak{X}) \rightarrow \Omega^0(M, u^*V\mathfrak{X})$ by

$$F\hat{u} := \sum_{i=1}^3 I(v_i) \bar{\nabla}_{v_i} \hat{u}$$

for some local orthonormal frame (v_1, v_2, v_3) . A simple computation yields

$$F\mathfrak{F}u = \bar{\nabla}^* \nabla u + \{\nabla u\}$$

where $\{\cdot\}$ makes the dependence on I etc. implicit. Further

$$\bar{\nabla} F\mathfrak{F}u = \bar{\nabla} \bar{\nabla}^* \nabla u + \{\nabla u\} + \{\bar{\nabla} \nabla u\}.$$

Using

$$\begin{aligned} \bar{\nabla}_{v_k} \bar{\nabla}_{v_i} \nabla_{v_i} u &= \bar{\nabla}_{v_i} \bar{\nabla}_{v_k} \nabla_{v_i} u + \{\nabla u, \nabla u, \nabla u\} \\ &= \bar{\nabla}_{v_i} \bar{\nabla}_{v_i} \nabla_{v_k} u + \{\nabla u, \nabla u, \nabla u\} + \{\bar{\nabla} \nabla u\} \end{aligned}$$

and $\mathfrak{F}u = \mathfrak{p} \circ u$ we derive

$$\begin{aligned} \bar{\nabla}^* \bar{\nabla} \nabla u &= \bar{\nabla} F\mathfrak{F}u + \{\nabla u, \nabla u, \nabla u\} + \{\bar{\nabla} \nabla u\} \\ &= \{\nabla u, \nabla u, \nabla u\} + \{\bar{\nabla} \nabla u\} + O(1). \end{aligned}$$

From this it follows that

$$\begin{aligned}
\Delta|\nabla u|^2 &= 2\langle \bar{\nabla}^* \bar{\nabla} \nabla u, \nabla u \rangle - 2|\bar{\nabla} \nabla u|^2 \\
&\leq c(|\nabla u|^4 + |\nabla u| + |\bar{\nabla} \nabla u||\nabla u|^2) - 2|\bar{\nabla} \nabla u|^2 \\
&\lesssim |\nabla u|^4 + 1.
\end{aligned}
\tag*{\square}$$

Remark 3.5. If $\mathfrak{X} = M \times X$ and X is flat, then one can prove that

$$\Delta|\nabla u|^2 \lesssim |\nabla u|^3 + 1$$

and the Heinz trick for subcritical exponents shows that $\|\nabla u\|_{L^\infty(M)}$ is bounded in terms of the energy $\mathcal{E}(u)$; see [Remark A.2](#) and [\[HNS09, Appendix B\]](#).

4 Convergence away from the blow-up locus

With ε_0 as in [Proposition 3.1](#) we define the following set on which convergence necessarily has to fail.

Definition 4.1. The *blow-up locus* of the sequence (u_i) is

$$S = S[(u_i)] := \left\{ x \in M : \liminf_{i \rightarrow \infty} \frac{1}{r} \int_{B_r(x)} |\nabla u_i|^2 \geq \varepsilon_0 \text{ for all } r \in (0, r_0] \right\}.$$

Proposition 4.2. *S is closed and satisfies $\mathcal{H}^1(S) < \infty$. Moreover, there exists a section $u \in \Gamma(M \setminus S, \mathfrak{X})$ such that (after passing to a subsequence) $u_i|_{M \setminus S}$ converges to u in C_{loc}^∞ .*

Proof. Let $x \in M \setminus S$. Then for some $0 < r \leq r_0$ (after passing to a subsequence) we have

$$\frac{1}{r} \int_{B_r(x)} |\nabla u_i|^2 < \varepsilon_0$$

for all i . Therefore, $|\nabla u_i|$ is uniformly bounded on $B_{r/4}(x)$. It follows that $B_{r/8}(x) \subset M \setminus S$; hence, S is closed. It also follows using standard elliptic techniques and Arzelà–Ascoli that a subsequence of u_i converges in C_{loc}^∞ on $M \setminus S$.

To see that $\mathcal{H}^1(S) < \infty$, given $0 < \delta \leq r_0$, cover S by a collection of balls $\{B_{2r_j}(x_j) : j = 1, \dots, m\}$ with $x_j \in S$, $r_j \leq \delta$ and $B_{r_j}(x_j)$ pairwise disjoint. For $i \gg 1$,

$$\frac{1}{r_j} \int_{B_{r_j}(x_j)} |\nabla u_i|^2 \geq \frac{\varepsilon_0}{2}.$$

Hence,

$$\sum_{j=1}^m r_j \leq \frac{2}{\varepsilon_0} \sum_{j=1}^m \int_{B_{r_j}(x_j)} |\nabla u_i|^2 \leq \frac{2}{\varepsilon_0} \int_M |\nabla u_i|^2 \leq \frac{2c_{\mathcal{E}}}{\varepsilon_0} < \infty. \tag*{\square}$$

5 Decomposition of the blow-up locus

We assume that we have already passed to a subsequence so that the convergence statement in [Proposition 4.2](#) holds. Consider the sequence of measures (μ_i) defined by

$$\mu_i := |\nabla u_i|^2 \mathcal{H}^3.$$

Here \mathcal{H}^3 is the 3-dimensional Hausdorff measure on M , which is simply the standard measure on M . By [\(1.11\)](#) the sequence of Radon measures (μ_i) is of bounded mass; hence, it converges weakly to a Radon measure μ . By Fatou's lemma we can write

$$\mu = |\nabla u|^2 \mathcal{H}^3 + \nu$$

for some non-negative Radon measure ν .

Definition 5.1. We call ν the *defect measure* and

$$\Gamma := \text{supp } \nu$$

the *bubbling locus*.⁶ We call

$$\text{sing}(u) := \left\{ x \in M : \Theta_u^*(x) := \limsup_{r \downarrow 0} \frac{1}{r} \int_{B_r(x)} |\nabla u|^2 > 0 \right\}$$

the *singular set* of u .

If we denote by $\Theta_\mu^*(x)$ the upper density of μ at the point $x \in M$, then it follows from [Proposition 3.1](#) that $S = \{x \in M : \Theta_\mu^*(x) > 0\} \subset \Gamma \cup \text{sing}(u)$. The reverse inclusion also holds; hence, we have the following.

Proposition 5.2. *The blow-up locus S decomposes as*

$$S = \Gamma \cup \text{sing}(u).$$

This means that there are two sources of non-compactness: one involving a loss of energy and another one without any loss of energy.

6 Regularity of the bubbling locus

As a first step towards understanding the non-compactness phenomenon involving energy loss, we show that the set Γ at which this phenomenon occurs is relatively tame.

⁶The justification for this terminology will be provided in [Section 7](#).

Proposition 6.1. Γ is \mathcal{H}^1 -rectifiable and ν can be written as

$$\nu = \Theta \mathcal{H}^1 \llcorner \Gamma$$

with $\Theta: M \rightarrow [0, \infty)$ upper semi-continuous. Moreover, $\mathcal{H}^1(\text{sing}(u)) = 0$.

The interested reader can find a detailed discussion of the concept of rectifiability in DeLellis' lecture notes [DL08]. For our purposes it shall suffice to recall the definition.

Definition 6.2. A subset $\Gamma \subset M$ is called \mathcal{H}^k -rectifiable if there exists a countable collection $\{\Gamma_i\}$ of k -dimensional Lipschitz submanifolds such that

$$\mathcal{H}^k\left(\Gamma \setminus \bigcup_i \Gamma_i\right) = 0.$$

A measure μ on M is called \mathcal{H}^k -rectifiable if there exist a non-negative Borel measurable function Θ and \mathcal{H}^k -rectifiable set Γ such that for any Borel set A

$$\mu(A) = \int_{A \cap \Gamma} \Theta(x) \mathcal{H}^k.$$

Since Γ is \mathcal{H}^1 -rectifiable, at \mathcal{H}^1 -a.e. point $x \in \Gamma$, it has a well-defined tangent space $T_x \Gamma$ and ν has a *tangent measure*, i.e., the limit

$$T_x \nu := \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} (\exp \circ s_\varepsilon)^* \nu$$

exists and

$$T_x \nu = \Theta(x) \mathcal{H}^1 \llcorner T_x \Gamma.$$

Here $s_\varepsilon(x) := \varepsilon x$.

To prove **Proposition 6.1** we will make use of the following deep theorem, whose proof is carefully explained in [DL08].

Theorem 6.3 (D. Preiss [Pre87]). *If μ is a locally finite measure on M and $m \in \mathbf{N}_0$ is such that for μ -a.e. $x \in M$ the density*

$$\Theta_\mu^m(x) := \lim_{r \downarrow 0} \frac{\mu(B_r(x))}{r^m}.$$

exists and is finite, then μ is \mathcal{H}^m -rectifiable.

*Proof of **Proposition 6.1**.* The proof has five steps.

Step 1. With the same constant as in [Proposition 2.1](#) and for all $x \in M$ and $0 < s \leq r$

$$e^{cs}s^{-1}\mu(B_s(x)) \leq e^{cr}r^{-1}\mu(B_r(x)) + cr^2.$$

This is not quite a trivial consequence of [Proposition 2.1](#) because (μ_i) only weakly converges to μ ; hence, we only know that $\mu(\bar{B}_r(x)) \geq \limsup_{i \rightarrow \infty} \mu_i(\bar{B}_r(x))$ and $\liminf_{i \rightarrow \infty} \mu_i(B_r(x)) \geq \mu(B_r(x))$.

For $x \in M$ set

$$\mathcal{R}_x := \{r \in (0, r_0] : \mu(\partial B_r(x)) > 0\}.$$

If $r \notin \mathcal{R}_x$, then it follows from [Proposition 2.1](#) that

$$e^{cs}s^{-1}\mu(B_s(x)) \leq e^{cr}r^{-1}\mu(B_r(x)) + cr^2.$$

The general case follows by an approximation argument. Note that \mathcal{R}_x is at most countable. Thus, given $r \in \mathcal{R}_x$, we can find a sequence (r_i) such that $s < r_i < r$, $r_i \notin \mathcal{R}_x$, and $r := \lim_{i \rightarrow \infty} r_i$. By dominated convergence

$$\mu(B_r(x)) = \lim_{i \rightarrow \infty} \mu(B_{r_i}(x)).$$

Step 2. *The limit*

$$\Theta(x) := \lim_{r \downarrow 0} r^{4-n} \mu(B_r(x))$$

exists for all $x \in M$. The function $\Theta: M \rightarrow [0, \infty)$ is upper semi-continuous, it vanishes outside S , is bounded and $\Theta(x) \geq \varepsilon_0$ for all $x \in S$.

The existence of the limit is a direct consequence of [Step 1](#).

To see that Θ is upper semi-continuous, let (x_i) be a sequence of points in M converging to a limit point $x = \lim_{i \rightarrow \infty} x_i$. Let $r \notin \mathcal{R}_x$ and $\varepsilon > 0$. For $i \gg 1$

$$\Theta(x_i) \leq e^{cr}r^{-1}\mu(B_r(x_i)) + cr^2 \leq e^{cr}r^{-1}\mu(B_{r+\varepsilon}(x)) + cr^2.$$

Therefore, $\limsup_{i \rightarrow \infty} \Theta(x_i) \leq e^{cr}r^{-1}\mu(B_r(x)) + cr^2$. Taking the limit as $r \rightarrow 0$ shows that Θ is upper semi-continuous.

The last part is clear.

Step 3. Θ_u^* vanishes \mathcal{H}^1 -a.e. in M , i.e., $\mathcal{H}^1(\text{sing}(u)) = 0$.

Given $\varepsilon > 0$, set

$$E_\varepsilon := \{x \in M : \Theta_u^*(x) > \varepsilon\}.$$

Given $0 < \delta$, choose $\{x_1, \dots, x_m\} \subset E_\varepsilon$ and $\{r_1, \dots, r_m\} \subset (0, \delta]$ such that the balls $B_{2r_j}(x_j)$ cover E_ε , but the balls $B_{r_j}(x_j)$ are pairwise disjoint. Moreover, we can arrange that

$$\frac{1}{r_j} \int_{B_{r_j}(x_j)} |\nabla u|^2 > \varepsilon.$$

Since u is smooth on $M \setminus S$, we must have $E_\varepsilon \subset S$. Hence,

$$\sum_{j=1}^m r_j \leq \frac{1}{\varepsilon} \sum_{j=1}^m \int_{B_{r_j}(x_j)} |\nabla u|^2 \leq \frac{1}{\varepsilon} \sum_{j=1}^m \int_{N_\delta(S)} |\nabla u|^2$$

where $N_\delta(S) = \{x \in M : d(x, S) < \delta\}$. The right-hand side goes to zero as δ goes to zero. Thus $\mathcal{H}^1(E_\varepsilon) = 0$ for all $\varepsilon > 0$. This concludes the proof.

Step 4. ν is \mathcal{H}^1 -rectifiable.

By [Step 2](#) for any $x \in M \setminus \text{sing}(u)$ the density

$$\Theta_\nu(x) = \lim_{r \downarrow 0} \frac{\nu(B_r(x))}{r}$$

exists and agrees with $\Theta(x)$. By [Step 3](#), $\mathcal{H}^1(\text{sing}(u)) = 0$ and, hence, $\nu(\text{sing}(u)) = 0$. Applying [Theorem 6.3](#) yields the assertion.

Step 5. We prove the proposition.

We have already proved the assertion about $\text{sing}(u)$. Since ν is \mathcal{H}^1 -rectifiable and $\Gamma = \text{supp}(\nu)$, it follows that Γ is \mathcal{H}^1 -rectifiable and ν can be written as

$$\nu = \Theta_\nu \mathcal{H}^1 \llcorner \Gamma$$

for some $\tilde{\Theta}$. By [Step 3](#), $\Theta_\nu(x) = \Theta(x)$ for \mathcal{H}^1 -a.e. $x \in \Gamma$. □

7 Bubbling analysis

We will now show that the “lost energy” goes into the formation of bubbles transverse to Γ . To state the main result recall that an orientation on $N_x \Gamma$ induces a canonical complex structure and an orientation of $N_x \Gamma$ is canonically determined by the choice of a unit tangent vector $v \in T_x \Gamma \subset T_x M$ since M is oriented.

Proposition 7.1. *If $x \in \Gamma$ is smooth, i.e., $T_x\Gamma$ exists and $x \notin \text{sing}(u)$, then there exists a $I(v)$ -holomorphic sphere $\mathfrak{z}_x : N_x\Gamma \cup \{\infty\} \rightarrow X := \mathfrak{X}_x$ with*

$$(7.2) \quad \mathcal{E}(\mathfrak{z}_x) := \int_{S^2} |\mathrm{d}\mathfrak{z}_x|^2 \leq \Theta(x).$$

Here we have picked some unit vector $v \in T_x\Gamma$.

Remark 7.3. It is immaterial whether we choose v or its opposite $-v$ since this results in changing the complex structures on both $N_x\Gamma$ and X . In particular, the above cannot be used to fix an orientation of Γ ; however, the existence of \mathfrak{z}_x does restrict the possible tangent directions, see [Section 8](#).

Remark 7.4. The reason that (7.2) may be strict is that we only extract one bubble of what is an entire bubbling-tree, cf. Parker–Wolfson [PW93]. Constructing an entire bubbling tree, however, requires a significant amount of work and additional insight to make sure that no energy is lost on necks between bubbles and that bubbles connect. In the related problem of harmonic maps into spheres the construction of the bubbling-tree was carried out by Lin–Rivière [LR02].

The holomorphic sphere \mathfrak{z}_x is obtained by blowing-up (u_i) around the point $x \in \Gamma$. We assume a trivialisation of \mathfrak{X} in a neighbourhood U of x has been fixed; see [Example 1.5](#). We use the following notation: given any map $u : U \rightarrow X$ and a scale factor $0 < \lambda$, we define a rescaled map $u_\lambda : B_{r_0/\lambda}^3(0) \rightarrow X$ by

$$(7.5) \quad u_\lambda := u(\exp \circ s_\lambda).$$

with $s_\lambda(y) := \lambda y$. We write (z, w) to denote points in $T_x\Gamma \times N_x\Gamma = T_xM$ and work with generalised cubes of the form

$$Q_{r,s}(z_0, w_0) := B_r(z_0) \times B_s(w_0) \subset T_x\Gamma \times N_x\Gamma = T_xM.$$

Proof of [Proposition 7.1](#). We proceed in three steps.

Step 1 (Preliminary scale fixing). *There exists a null-sequence $(\varepsilon_i) \subset (0, 1)$ such that*

$$|\mathrm{d}u_{i;\varepsilon_i}|^2 \mathcal{H}^3 \rightharpoonup T_x\nu = \Theta(x) \mathcal{H}^1 \llcorner T_x\Gamma.$$

By definition, $T_x\nu$ is the weak limit of $(\exp \circ s_\varepsilon)^*\nu$ as ε tends to zero. Since $x \notin \text{sing}(u)$, we have

$$\lim_{\varepsilon \rightarrow 0} (\exp \circ s_\varepsilon)^*\nu = \lim_{\varepsilon \rightarrow 0} (\exp \circ s_\varepsilon)^*\mu.$$

Thus

$$T_x\nu = \lim_{\varepsilon \rightarrow 0} \lim_{i \rightarrow \infty} (\exp \circ s_\varepsilon)^*\mu_i = \lim_{i \rightarrow \infty} (\exp \circ s_{\varepsilon_i})^*\mu_i$$

for some null-sequence (ε_i) . This implies the assertion since

$$(\exp \circ s_{\varepsilon_i})^* \mu_i = |du_{i;\varepsilon_i}|^2 \mathcal{H}^3.$$

Step 2 (Asymptotic translation invariance). *For any sequence $(\delta_i) \subset (0, 1]$ and \mathcal{H}^1 -a.e. $z \in T_x \Gamma$*

$$(7.6) \quad \lim_{i \rightarrow \infty} \sup_{s \leq 1/\delta_i} \frac{1}{s} \int_{Q_{s,1/\delta_i}(z,0)} |\partial_v u_{i;\delta_i \varepsilon_i}|^2 = 0.$$

It suffices to prove this for $\delta_i = 1$ and $z \in B_1(0)$, because the assertion is invariant under rescaling and unaffected by translating everything along the direction of $T_x \Gamma$.

Step 2.1. *We have*

$$\lim_{i \rightarrow \infty} \int_{Q_{2,1}(0)} |\partial_v u_{i,\varepsilon_i}|^2 = 0.$$

Denote by ∂_ρ the radial vector field emanating from $4v$. By [Proposition 2.1](#), for for $0 < s \leq r$

$$(7.7) \quad \begin{aligned} & \int_{B_r(4v) \setminus B_s(4v)} e^{c\varepsilon_i \tau} \tau^{-1} |\partial_\rho u_{i;\varepsilon_i}|^2 \\ & \leq e^{c\varepsilon_i r} r^{-1} \int_{B_r(4v)} |du_{i;\varepsilon_i}|^2 - e^{c\varepsilon_i s} s^{-1} \int_{B_s(4v)} |du_{i;\varepsilon_i}|^2 + c\varepsilon_i^2 r^2. \end{aligned}$$

As i tends to infinity the first two terms on the right-hand side both converge to $\Theta(x)$, since $T_x \nu = \Theta(x) \mathcal{H}^1|_{T_x \Gamma}$ and the last term tends to zero.

Since $Q_{2,1}(0) \subset B_8(4v) \setminus B_1(4v)$, it follows that

$$\lim_{i \rightarrow \infty} \int_{Q_{2,1}(0)} |\partial_\rho u_{i,\varepsilon_i}|^2 = 0.$$

This completes the proof, because along $T_x \Gamma \cap B_2(0)$ the vector fields ∂_ρ and v are colinear and $|\partial_v u_{i,\varepsilon_i}|^2 \mathcal{H}^3$ converges to zero outside $T_x \Gamma$.

Step 2.2. *We prove (7.6).*

Define $f_i: B_2(0) \subset T_x \Gamma \rightarrow [0, \infty)$ by

$$f_i(z) := \int_{B_1(0) \subset N_x \Gamma} |\partial_v u_{i;\varepsilon_i}|^2(z, \cdot)$$

and denote by $Mf_i: B_1(0) \subset T_x\Gamma \rightarrow [0, \infty)$ the Hardy–Littlewood maximal function associated with f_i :

$$Mf_i(z) := \sup_{s \leq 1} \frac{1}{s} \int_{B_s(z) \subset T_x\Gamma} f_i.$$

By the weak-type L^1 estimate for the maximal operator for each $\delta > 0$

$$\mathcal{H}^1(\{z : Mf_i(z) \geq \delta\}) \lesssim \frac{\|f_i\|_{L^1}}{\delta}.$$

Since $\|f_i\|_{L^1} \rightarrow 0$ by the previous step, the assertion follows.

Step 3. We prove *Proposition 7.1*.

By *Step 1*,

$$\liminf_{i \rightarrow \infty} \max_{w \in B_1(0) \subset N_x\Gamma} \frac{1}{\delta} \int_{Q_{\delta,\delta}(z_i,w)} |du_{i;\varepsilon_i}|^2 = \Theta(x) \geq \varepsilon_0$$

for all $\delta > 0$, while for fixed $i \in \mathbb{N}$ and $w \in B_1(0) \subset N_x\Gamma$

$$\lim_{\delta \downarrow 0} \frac{1}{\delta} \int_{Q_{\delta,\delta}(z_i,w)} |du_{i;\varepsilon_i}|^2 = 0.$$

Hence, we can find a null sequence (δ_i) such that

$$\max_{w \in B_{1/\delta_i}(0) \subset N_x\Gamma} \frac{1}{\delta_i} \int_{B_{\delta_i}(z_i,w)} |du_{i;\varepsilon_i}|^2 = \varepsilon_0/8.$$

Define

$$\tilde{u}_i := \tilde{u}_i(\cdot) := u_{i;\delta_i\varepsilon_i}(\delta_i(z_i, w_i) + \cdot).$$

By construction

$$\max_{w \in B_1(0) \subset N_x\Gamma} \int_{B_1(0,w)} |d\tilde{u}_i|^2 = \varepsilon_0/8$$

with the maximum achieved at $w = 0$.

Let $\phi: T_xM \rightarrow [0, \infty)$ denote a smooth function, which is compactly supported in $B_1(0)$ and equal to one on $B_{1/2}(0)$. Define $e_i: B_{(r_0/\delta_i\varepsilon_i)-1}(0) \rightarrow [0, \infty)$ by

$$e_i(\zeta, \omega) := \int \phi(\cdot) \cdot |d\tilde{u}_i|^2((\zeta, \omega) + \cdot).$$

By integration by parts, for $v \in T_x\Gamma$,

$$|\partial_v e_i(\zeta, \omega)| \lesssim \|\phi\|_{C^1} \int_{B_1(\zeta, \omega)} |\nabla \tilde{u}_i| (|\nabla_v \tilde{u}_i| + o(1)).$$

The $o(1)$ -term accounts for the connection on \mathfrak{X} being non-flat. By [Step 2](#), $|\partial_v e_i(\zeta, \omega)|$ goes to zero uniformly on compact subsets of $T_x M$. This means that, for each $z \in T_x \Gamma$, $w \in B_{1/\delta_i}(0) \subset N_x \Gamma$ and $i \gg_z 1$,

$$\int_{B_{1/2}(z, w)} |d\tilde{u}_i|^2 \leq \varepsilon_0/4.$$

From [Proposition 3.1](#) and [Remark 3.3](#) we obtain C_{loc}^∞ -bounds on \tilde{u}_i which allow us to pass to a limit $u: T_x M \rightarrow X$, which solves the Fueter equation. By [Step 2](#), u is invariant under translations along $T_x \Gamma$ and, hence, the pullback of a map $\mathfrak{z}: N_x \Gamma \rightarrow X$. We can choose the orthonormal frame (v_1, v_2, v_3) on $T_x M$ constant and with $v_1 = v \in T_x \Gamma$ and $v_2, v_3 \in N_x \Gamma$. With respect to this frame the Fueter operator takes the form

$$\mathfrak{F} = I(v_1)\partial_v + I(v_2)\bar{\partial}$$

with $\bar{\partial} = \partial_{v_2} - I(v)\partial_{v_3}$. Thus \mathfrak{z} is $I(v)$ -holomorphic. \square

Question 7.8. What happens near non-smooth points of Γ ?

8 Constraints on tangent directions

By [Proposition 7.1](#), if $v \in ST_x \Gamma$, then \mathfrak{X}_x must admit a non-trivial $I(v)$ -holomorphic sphere \mathfrak{z}_x of area at most $\Theta(x)$. Since Θ is upper semi-continuous, it achieves a maximum A_{max} on Γ . Thus, the area of \mathfrak{z}_x is bounded by A_{max} and the following shows that the possible tangent directions of Γ are strongly constrained.

Proposition 8.1. *Let X be a simple hyperkähler manifold with $b_2(X) \geq 6$. Given $A_{\text{max}} > 0$, there exists only finitely many $I_\xi \in \mathfrak{H}(X)$ for which there exists a rational curve C in (X, I_ξ) with*

$$\text{area}(C) = \langle [C], \omega_\xi \rangle \leq A_{\text{max}}.$$

Here $\omega_\xi = g(I_\xi, \cdot, \cdot)$.

If X is a $K3$ surface, then this is essentially contained in Bryan–Leung [[BL00](#), Proposition 3.1]. Its proof mainly uses some facts about the $K3$ -lattice $(H^2(K3, \mathbf{Z}), \cup)$. The appropriate replacement of the cup-product for general simple hyperkähler manifold is the *Beauville–Bogomolov–Fujiki (BBF) form* $q: S^2 H^2(X, \mathbf{Z}) \rightarrow \mathbf{Z}$. We refer the reader to [[Bea83](#), [Bog78](#), [Fuj87](#)] for details about the BBF form. For our purposes it suffices to recall that:

- q is non-degenerate, i.e., the induced map $H^2(X, \mathbf{Q}) \rightarrow H^2(X, \mathbf{Q})^*$ is an isomorphism. In particular, for each $C \in H_2(X, \mathbf{Z})$ there exists a unique $\gamma \in H^2(X, \mathbf{Q})$ such that

$$(8.2) \quad q(\gamma, \cdot) = \langle C, \cdot \rangle \in H^2(X, \mathbf{Q})^*.$$

- q has signature $(3, b_2(X) - 3)$ with $\text{span} \{[\omega_\xi] : \xi \in S^2\}$ forming a maximal positive definite subspace. We denote the perpendicular maximal negative definite subspace by N .

Theorem 8.3 (Amerik–Verbitsky). *If X is a simple hyperkähler manifold with $b_2(M) \geq 6$, then there exists a positive integer $\sigma \in \mathbf{N}$ such that*

$$q(\gamma, \gamma) \geq -\sigma$$

for all $\gamma \in H^2(X, \mathbf{Q})$ with (8.2) for some C represented by a I_ξ -holomorphic sphere for some $I_\xi \in \mathfrak{H}(X)$.

Proof. This follows by observing that γ is a MBM class in the sense of [AV14, Definition 2.14] and then appealing to [AV14, Theorem 5.3]. \square

Remark 8.4. **Theorem 8.3** generalises the fact that any class representing a holomorphic sphere in $K3$ has square -2 .

Proposition 8.5. *There exists a constant $c_0 > 0$ such that if C is represented by a I_ξ -holomorphic sphere of area A , then γ as in (8.2) is of the form*

$$(8.6) \quad \gamma = \beta + c_0 A \omega_\xi$$

with $\beta \in N$ and

$$q(\beta, \beta) \geq -\sigma - c_0 A^2.$$

Proof. It follows from (8.2) that

$$(8.7) \quad q(\gamma, \omega_\eta) = 0$$

for all $\eta \perp \xi$; hence, $\gamma = \beta + c_0 A \omega_\xi$ with $c_0 = 1/q(\omega_\xi, \omega_\xi)$, which does not depend on $\xi \in S^2$, and $\beta \in N$. Since $q(\gamma, \gamma) \geq -\sigma$, we have

$$q(\beta, \beta) \geq -\sigma - c_0 A_{\max}^2. \quad \square$$

Proof of Proposition 8.1. There are only finitely many γ as in Proposition 8.5 with $A \leq A_{\max}$ and γ determines $\xi \in S^2$ uniquely. \square

9 The singular set of u

Proposition 9.1. *The monotonicity formula, i.e., [Proposition 2.1](#), holds for u .*

Repeating the arguments used earlier immediately establishes the following.

Corollary 9.2. $\Theta_u := \Theta_u^* : M \rightarrow [0, \infty)$ is upper semi-continuous and $\text{sing}(u)$ is closed.

Proof of [Proposition 9.1](#). The proof of [Proposition 2.1](#) uses the fact that u is smooth on all of M only in the application of Stokes' theorem in [\(2.9\)](#), i.e., to show that

$$\int_{B_\rho(x)} u^* d\Omega = \int_{\partial B_\rho(x)} u^* \Omega.$$

To see that this still holds in the current situation, fix a cut-off function χ which is zero on $[0, 1]$ and one on $[2, \infty]$, and set

$$\chi_r(x) := \chi(d(x, S)/r).$$

By the dominated convergence theorem

$$\begin{aligned} \int_{B_\rho(x)} u^* d\Omega &= \lim_{r \downarrow 0} \int_{B_\rho(x)} \chi_r u^* d\Omega \\ &= \int_{\partial B_\rho(x)} u^* \Omega - \lim_{r \downarrow 0} \int_{B_\rho(x)} d\chi_r \wedge u^* \Omega. \end{aligned}$$

To see that the last term vanishes, recall that Ω is of type $(0, 2)$ and $|\Omega| = O(\underline{r})$. Therefore, $|u^* \Omega| \lesssim r |\nabla u|^2$. Since $|d\chi_r| \lesssim 1/r$, the integral in the limit is bounded by a constant times

$$\int_{\text{supp } d\chi_r} |\nabla u|^2,$$

which converges to zero as r goes to zero. \square

In the following section we will repeatedly use integration by parts as above. Each application can be justified in the same way as above.

10 Tangent cones

To further analyse $\text{sing}(u)$ it is customary to study tangent cones.

Definition 10.1. Let $x \in M$ and let (r_i) be a null-sequence. Let u_{r_i} be the rescaling of u centered at x ; see (7.5). After passing to a subsequence this converges to a limit $(u_*; \Gamma_*, \Theta_*)$ with $\Gamma_* \subset \mathbf{R}^3$ a 1-dimensional rectifiable set, $u_*: \mathbf{R}^3 \setminus \Gamma_* \rightarrow X$ a Fueter map and $\Theta_*: \mathbf{R}^3 \rightarrow [0, \infty)$ an upper semi-continuous map supported on Γ_* . Such a limit is called a *tangent cone* to u at x .

Remark 10.2. If $x \notin \text{sing}(u)$, then u_* is the constant map $u(x)$ and $\Gamma_* = \emptyset$; hence, it is only interesting to consider tangent cones to $x \in \text{sing}(u)$.

Question 10.3 (Uniqueness of tangent cones). Is $(u_*; \Gamma_*, \Theta_*)$ independent of the sequence (r_i) ?

Proposition 10.4. Every tangent cone $(u_*; \Gamma_*, \Theta_*)$ is conical: for all $r > 0$,

$$s_r^*(u_*; \Gamma_*, \Theta_*) = (u_*; \Gamma_*, \Theta_*).$$

Combing this with $\mathcal{H}^1(\text{sing}(u_*)) = 0$ yields the following.

Corollary 10.5. For every tangent cone $(u_*; \Gamma_*, \Theta_*)$, $\text{sing}(u_*) \subset \{0\}$.

Remark 10.6. Proposition 10.4 can be viewed as an affirmative answer to the weaker version of Question 10.3: is $(u_*; \Gamma_*, \Theta_*)$ invariant under rescaling (r_i) to $(r \cdot r_i)$?

The proof relies on the following estimate.

Proposition 10.7. Let $U \subset \mathbf{R}^3$ be an open subset and $u \in \Gamma(U, \mathfrak{X})$ be a solution of (1.10). Let $\phi \in C_c^1(\mathbf{R}^3)$ and $R \geq 1$ such that $[1, R] \cdot \text{supp}(\phi) \subset U$. Then

$$(10.8) \quad \left| \int \phi(x) |\nabla u|^2(Rx) - \int \phi(x) |\nabla u|^2(x) \right| \\ \lesssim \int_{r=1}^R \int_{\text{supp} \phi} (\text{diam supp } \phi) \|\phi\|_{C^1} |\nabla_r u| |\nabla u|(rx) \\ + \phi(x) |\mathfrak{p} \circ u|^2(Rx) + |Rx|^2 \phi(x) \delta |\nabla u|^2(rx)$$

with $\delta = |\nabla I| + |F_{\mathfrak{X}}| + |R|$ as in (2.8).

Proof. Let us first assume that U is flat, I is parallel and u solves (1.6). The derivative of

$$f(R) := \int \phi(x) |du|^2(Rx).$$

is

$$f'(R) = \int \phi(x) |x| \partial_r (|du|^2(Rx)) = - \int \partial_r (|x| \phi) |du|^2(Rx) \\ = - \int \phi(x) |du|^2(Rx) + |x| (\partial_r \phi) |du|^2(Rx).$$

Using the identities (2.4) and (2.6) we obtain

$$\begin{aligned}
(10.9) \quad - \int \phi |du|^2 &= -2 \int \phi \sum_{i=1}^3 dx_i \wedge u^* \omega_i \\
&= -2 \int \phi d(|x| u^* \omega_{\partial_r}) \\
&= 2 \int |x| d\phi \wedge u^* \omega_{\partial_r} \\
&= \int |x| (\partial_r \phi) (|du|^2 - 2|\partial_r u|^2) \\
&\quad + 2|x| (\partial_1 \phi) \omega_{\partial_r} (\partial_2 u, \partial_r u) + |x| (\partial_2 \phi) \omega_{\partial_r} (\partial_r u, \partial_1 u).
\end{aligned}$$

In the expression for $f'(R)$ the contributions from $\phi(x)|du|^2(Rx)$ cancel and thus

$$(10.10) \quad |f'(R)| \leq (\text{diam supp } \phi) \|\phi\|_{C^1} \int_{\text{supp } \phi} |\partial_r u| |du|(Rx).$$

This yields the desired estimate by integration.

In general, since u only solves the perturbed Fueter equation, when using (2.7) and (2.6) in the first and the last step of (10.9) we acquire the additional terms

$$\int \phi(x) |\mathfrak{p} \circ u|^2(Rx) + \partial_r(|x|\phi) |\mathfrak{p} \circ u|^2(Rx)$$

Moreover, in the second step of (10.9) the additional error term

$$\int \phi(x) O(|Rx|^2 \delta).$$

appears. □

Proof of Proposition 10.4. Applying Proposition 10.7 along u_{r_i} shows that the measure

$$\mu_* = |du_*|^2 \mathcal{H}^3 + \Theta_* \mathcal{H}^1 \llcorner \Gamma_*$$

is conical, since terms on the right-hand side of (10.8) tend to zero as $r_i \rightarrow 0$. This implies that Γ_* and Θ_* are conical. Moreover, the monotonicity formula shows that $\nabla_r u_* = 0$; hence, u_* is conical. □

10.1 Filtration by tangent cones

Following Simon [Sim96, Section 3.4] we define a filtration $S_0 \subset S_1 = \text{sing}(u)$ according to the maximal number of lines a non-trivial tangent cone at x splits off:

$$S_k := \{x \in \text{sing}(u) : \text{all tangent cones to } u \text{ at } x \text{ split off at most } k \text{ lines}\}$$

Here we say that a tangent cones splits of k lines if for some orthogonal splitting $\mathbf{R}^k \oplus \mathbf{R}^{3-k}$ it is the pullback from \mathbf{R}^{3-k} .

Remark 10.11. In our situation a non-trivial tangent cone can split off at most one line. This is why the filtration has only two steps.

The significance of this filtration is that $\dim S_k \leq k$; see [Sim96, Section 3.4, Lemma 1].

Proposition 10.12. *If $(u_*; \Gamma_*, \Theta_*)$ is non-trivial and splits of a line $\mathbf{R} \in \mathbf{R}^3$, then $\Gamma_* = \mathbf{R}$, Θ_* is a positive constant and u_* is constant.*

Proof. Certainly we must have $\Gamma_* \subset \mathbf{R}$. The Fueter equation and \mathbf{R} -translation-invariance forces $u_*|_{\mathbf{R}^2 \setminus \{0\}} \rightarrow X$ to be holomorphic. Since it is also conical, it must be constant. Clearly, if $\Gamma_* \neq \emptyset$, then $\Gamma_* = \mathbf{R}$ and Θ_* is constant. \square

Question 10.13. Can the situation in Proposition 10.12 occur?

Question 10.14. Are there tangent cones with $\Gamma_* \neq \emptyset$?

If the answer to Question 10.13 is no, then it would follow that $\text{sing}(u)$ is at most 0-dimensional.

Remark 10.15. In a closely related situation Chen–Li [CL00, Step 1 in the proof of Theorem 4.3] essentially assert that the answer to Question 10.14 is negative. However, the author finds himself unable to follow the details of their argument.

10.2 Holomorphic sections of the twistor fibration

Tangent maps can be interpreted as follows. Let $\text{Tw}(X) := \mathfrak{H}(X) \times X \cong S^2 \times X$ denote the *twistor space* of X equipped with the tautological complex structure on $\text{Tw}(X)$, given at (ξ, x) by $(I_{S^2} \oplus I_\xi)$; see Hitchin–Karlhede–Lindström–Roček [HKLR87, Section 3.F]. The projection $\text{Tw}(X) \xrightarrow{\pi} S^2$ is called the *twistor fibration*.

Proposition 10.16. *Let $u: S^2 \rightarrow X$. The section $\text{id} \times u \in \Gamma(\text{Tw}(X))$ is holomorphic if and only if $x \mapsto u(x/|x|)$ is a Fueter map.*

10.3 Balancing of tangent cones

An interesting property of tangent cones is that they are balanced.

Proposition 10.17. *If $(u_*; \Gamma_*, \Theta_*)$ is a tangent cone, then*

$$\int_{S^2} x \cdot |\mathrm{d}u_*|^2 = 0 \quad \text{and} \quad \sum_{x \in S^2 \cap \Gamma_*} x \cdot \Theta_*(x) = 0.$$

Remark 10.18. The analogous statement for harmonic maps was proved by Lin–Rivière as [LR02, Theorem B(iii)].

The proof hinges on the following.

Proposition 10.19. *If $u \in \Gamma(M, \mathfrak{X})$ satisfies (1.10), then for all $x \in M$ and $v \in \Gamma(B_r(x), TM)$*

$$\begin{aligned} & \left| \int_{\partial B_r(x)} \langle v, \partial_r \rangle (2|\nabla u|^2 + |\nabla_r u|^2) - 2\langle \nabla_r u, \nabla_v u \rangle \right| \\ & \lesssim \int_{\partial B_r(x)} r \|\nabla v\|_{L^\infty} |\nabla u|^2 + |\mathfrak{p} \circ u|^2 + \int_{B_r(x)} \delta |\nabla u|^2 + |\nabla(\mathfrak{p} \circ u)|^2 \end{aligned}$$

with $\delta = |\nabla I| + |F_{\mathfrak{X}}| + |R|$ as in (2.8).

Proof. Again, we first assume that M is flat, $\mathfrak{X} = M \times X$, I is parallel and u satisfies (1.6). We can assume that v is constant, because of the first term on the right-hand side. Consider

$$f(x) = \int_{B_r(x)} |\mathrm{d}u|^2.$$

By integration by parts

$$\partial_v f(x) = \int_{B_r(x)} \partial_v |\mathrm{d}u|^2 = \int_{\partial B_r(x)} \langle v, \partial_r \rangle |\mathrm{d}u|^2.$$

On the other hand

$$f(x) = 2 \int_{B_r(x)} \sum_{i=1}^3 \mathrm{d}x^i \wedge u^* \omega_i;$$

hence,

$$\partial_v f(x) = 2 \int_{B_r(x)} \mathcal{L}_v \sum_{i=1}^3 \mathrm{d}x^i \wedge u^* \omega_i = 2 \int_{\partial B_r(x)} i(v) \sum_{i=1}^3 \mathrm{d}x^i \wedge u^* \omega_i.$$

Choose $(v_1, v_2, v_3) = (\partial_r, \partial_1, \partial_2)$ as in the proof of [Proposition 2.2](#). Then

$$\begin{aligned} \left(i(v) \sum_{i=1}^3 dx^i \wedge u^* \omega_i \right) (e_1, e_2) &= \langle I_v \partial_1 u, \partial_2 u \rangle + \langle I_1 \partial_2 u - I_2 \partial_1 u, \partial_v u \rangle \\ &= \langle I_v \partial_1 u, \partial_2 u \rangle + \langle \partial_r u, \partial_v u \rangle \end{aligned}$$

A slightly length computation shows that

$$\begin{aligned} \langle I_v \partial_1 u, \partial_2 u \rangle &= -\frac{1}{2} \langle v, \partial_r \rangle |du|^2 - \langle v, \partial_r \rangle \langle I_r \partial_1 u, \partial_2 \rangle \\ &= -\langle v, \partial_r \rangle |du|^2 - \langle v, \partial_r \rangle |\partial_r u|^2. \end{aligned}$$

Putting everything together yields

$$\int_{\partial B_r} \langle v, \partial_r \rangle (2|du|^2 + |\partial_r u|^2) = 2 \int_{\partial B_r} \langle \partial_r u, \partial_v u \rangle.$$

In general, one has to adapt the above argument as in the proof of [Proposition 2.1](#) and obtains further error terms. \square

Proof of [Proposition 10.17](#). Apply [Proposition 10.19](#) along the sequence u_{r_i} . The terms on the right-hand side converge to zero and, by [Proposition 9.1](#), so do the terms involving $\nabla_r u$. For all $v \in \Gamma(B_r(x), TM)$

$$\lim_{i \rightarrow \infty} \int_{S^2} \langle v, \partial_r \rangle |\nabla u_{r_i}|^2 = \int_{S^2} \langle v, x \rangle \cdot |du_*|^2 + \sum_{x \in S^2 \cap \Gamma_*} \langle v, x \rangle \cdot \Theta_*(x);$$

hence,

$$\int_{S^2} x \cdot |du_*|^2 + \sum_{x \in S^2 \cap \Gamma_*} x \cdot \Theta_*(x) = 0.$$

The same reasoning applied to u_* directly shows that

$$\int_{S^2} x \cdot |du_*|^2 = 0.$$

This concludes the proof. \square

11 Tangent cones to (Γ, Θ)

We conclude this article with a slight refinement of the regularity statement of Γ .

Definition 11.1. Let $x \in \Gamma$ and let (r_i) be a null-sequence. Set

$$\Gamma_i := (\exp \circ s_{r_i})^{-1} \Gamma \quad \text{and} \quad \Theta_i := (\exp \circ s_{r_i})^* \Theta.$$

After passing to a subsequence $\Theta_i \mathcal{H}^1 \llcorner \Gamma_i$ weakly converges to a measure of the form $\Theta_* \mathcal{H}^1 \llcorner \Gamma_*$, with $\Gamma_* \subset \mathbf{R}^3$ a 1-dimensional rectifiable set and $\Theta_*: \mathbf{R}^3 \rightarrow [0, \infty)$ an upper semi-continuous map supported on Γ_* . Such a pair (Γ_*, Θ_*) is called a *tangent cone* to (Γ, Θ) at x .

Again, it is important to ask [Question 10.3](#).

Using [Proposition 10.7](#) and [Proposition 10.19](#) one can show the following.

Proposition 11.2. *If (Γ_*, Θ_*) is a tangent cone to (Γ, Θ) , then Γ_* is the cone on the finite set $\Gamma_* \cap S^2$:*

$$\Gamma_* = \bigcup_{x \in \Gamma_* \cap S^2} [0, \infty) \cdot x.$$

Θ_* is constant on each $(0, \infty) \cdot x$. Moreover,

$$\sum_{x \in \Gamma_* \cap S^2} \Theta_*(x) \cdot x = 0.$$

This means, in particular, that Γ cannot suddenly end at a point and does not contain any kinks.

A The Heinz trick

Throughout we consider a bounded open subset $U \subset \mathbf{R}^n$ endowed with a smooth metric g which extends smoothly to \bar{U} . Implicit constants are allowed to depend on the geometry of U .

Lemma A.1 (E. Heinz [\[Hei55\]](#)). *Fix $d > 0$ and set*

$$q := \frac{2}{d} + 1.$$

Suppose $f: U \rightarrow [0, \infty)$ and $p, \delta \in \{0, 1\}$ are such that the following hold:

1. *We have*

$$\Delta f \lesssim f^q + f^p.$$

2. *If $B_s(y) \subset B_{r/2}(x) \subset U$, then*

$$s^{d-n} \int_{B_s(y)} f \lesssim r^{d-n} \int_{B_r(x)} f + \delta r^2.$$

Then there exists a constant $\varepsilon_0 > 0$ such that for all $B_r(x) \subset U$ with

$$\varepsilon = r^{d-n} \int_{B_r(x)} f \leq \varepsilon_0$$

we have

$$\sup_{y \in B_{r/4}(x)} f(y) \lesssim r^{-d} \varepsilon + ((1-p) + \delta) r^2.$$

Remark A.2 (Heinz trick in the subcritical case). If $n < d$,

$$\varepsilon \leq \varepsilon_0 \quad \text{whenever} \quad r \leq \left(\frac{\varepsilon_0}{\int_U f} \right)^{\frac{1}{d-n}}.$$

In particular, for all compact $K \subset U$, $\|f\|_{L^\infty(K)}$ is bounded a priori depending only on $\int_U f$ and $d(K, \partial U)$.

We use the following standard result; see [GT01, Theorem 9.20] or [HNS09, Proof of Theorem B.1].

Proposition A.3. *For all $B_r(x) \subset U$ and every smooth function $f: B_r(x) \rightarrow [0, \infty)$*

$$f(x) \lesssim r^{-n} \int_{B_r(x)} f \, \text{dvol} + r^2 \|\Delta f\|_{L^\infty}.$$

Proof of Lemma A.1. Define a function $\theta: B_{r/2}(x) \rightarrow [0, \infty)$ by

$$\theta(y) := \left(\frac{r}{2} - d(x, y) \right)^d f(y).$$

Since θ is non-negative and vanishes on the boundary of $B_{\frac{r}{2}}(x)$, it achieves its maximum

$$M := \max_{y \in B_{\frac{r}{2}}(x)} \theta(y)$$

in the interior of $B_{\frac{r}{2}}(x)$. We will derive a bound for M , from which the assertion follows at once.

Let y_0 be a point with $\theta(y_0) = M$, set

$$F := f(y_0)$$

and denote by

$$s_0 := \frac{1}{2} \left(\frac{r}{2} - d(x, y_0) \right)$$

half the distance from y_0 to the boundary of $B_{\frac{r}{2}}(x)$. Each $y \in B_{s_0}(y_0)$ has distance from the boundary of $B_{\frac{r}{2}}(x)$ at least s_0 ; hence,

$$f(y) \leq s_0^{-d} \theta(y) \leq s_0^{-d} \theta(y_0) \lesssim F.$$

Proposition A.3 applied to $B_s(y_0)$ together with (1) and the above bound yields

$$F \lesssim s^{-n} \int_{B_s(y_0)} f + s^2 (F^q + F^p)$$

for all $0 \leq s \leq s_0$. Combined with (2) this becomes

$$F \lesssim s^{-d} \varepsilon + s^2 (F^q + F^p + \delta),$$

which can be rewritten as

$$(A.4) \quad s^d F \lesssim \varepsilon + s^{d+2} (F^q + F^p + \delta).$$

This inequality will yield the desired bound on M . It is useful to make a case distinction.

Case 1. $F \leq 1$.

In this case a bound on M follows from simple algebraic manipulations. If $p = 0$ or $\delta = 1$, then (A.4) with $s = s_0$ yields

$$M = \theta(y_0) \lesssim s_0^d F \lesssim \varepsilon + s_0^{d+2} \leq \varepsilon + r^{d+2}.$$

If $p = 1$ and $\delta = 0$, this bound can be sharpened. (A.4) becomes

$$s^d F \leq \frac{c\varepsilon}{1 - cs^2}.$$

If $cs_0^2 \leq \frac{1}{2}$, then we obtain

$$M \lesssim s_0^d F \lesssim \varepsilon;$$

otherwise, setting $s := (2c)^{-\frac{1}{2}} \leq s_0$ yields

$$F \lesssim \varepsilon,$$

and thus $M \lesssim \varepsilon$.

Case 2. $F > 1$.

From (A.4) we derive

$$s^d F \lesssim \varepsilon + s^{d+2} F^q$$

for all $0 \leq s \leq s_0$. Set $t := t(s) = sF^{1/d}$. Then the above inequality can be expressed as

$$t^d(1 - ct^2) \leq c\varepsilon.$$

For sufficiently small $\varepsilon > 0$, the corresponding equation $t^d(1 - ct^2) = c\varepsilon$ has d small roots t_1, \dots, t_d , which are approximately $\pm(c\varepsilon)^{\frac{1}{d}}$, and two large roots. Since $t(0) = 0$ and by continuity, for each $s \in [0, s_0]$, $t(s)$ must be less than the smallest positive root; hence, $t(s) \lesssim \varepsilon^{\frac{1}{d}}$ for all $s \in [0, s_0]$. This finishes the proof. \square

B Compactness for Fueter maps with four dimensional source manifold

Proposition B.1. *Let V be a 4-dimensional Euclidean vector space, H a quaternionic vector space, $I: S\Lambda^+V^* \rightarrow SH$ an isometric identification of the unit length self-dual forms on V with the unit quaternions and $\iota: \Lambda^+V^* \rightarrow \mathfrak{so}(V)$. The endomorphism $\Psi \in \text{End}(\text{Hom}(V, H))$ defined by*

$$\Psi T := \sum_{i=1}^3 I(\omega_i) \circ T \circ \iota(\omega_i)$$

has eigenvalues 1 and -3 . Here we sum over an orthonormal basis $(\omega_1, \omega_2, \omega_3)$ of Λ^+V^ . We denote the (-3) -eigenspace by $\text{Hom}_I(V, H)$.*

Let M be an orientable Riemannian 4-manifold, let $\mathfrak{X} \xrightarrow{\pi} M$ be a bundle of hyperkähler manifolds together with a fixed identification $I: S\Lambda^+T^*M \rightarrow \mathfrak{H}(\mathfrak{X})$ of the unit sphere bundle of self-dual forms on M and the bundle of hyperkähler spheres of the fibres of \mathfrak{X} and fix a connection on \mathfrak{X} .

Definition B.2. A section $u \in \Gamma(\mathfrak{X})$ is called a *Fueter section* if

$$(B.3) \quad \mathfrak{F}u := \nabla u - \Psi \nabla u = 0 \in \Gamma(u^* \text{Hom}_I(\pi^*TM, V\mathfrak{X})).$$

Remark B.4. If $M = \mathbf{R} \times N$ for some 3-manifold N , \mathfrak{X} is the pullback of a bundle \mathfrak{Y} of hyperkähler manifolds on N , I is obtained from an identification $J: STM \cong \mathfrak{H}(\mathfrak{X})$ and the connection on \mathfrak{X} is the pullback of a connection on \mathfrak{Y} , then (B.3) can be written as

$$\partial_t u - \mathfrak{F}u = 0$$

with \mathfrak{F} denoting the 3-dimensional Fueter operator. This is the form in which the 4-dimensional Fueter operator appears in [HNS09].

Remark B.5. Unlike in the 3-dimensional case, $\Lambda^+ T^* M$ need not be trivial.⁷ Thus the analogue of the setup in [Example 1.5](#) rarely makes sense globally, and one is almost forced to work with bundles of hyperkähler manifolds.

The analogue of [Theorem 1.9](#) in the 4-dimensional case is the following result.

Theorem B.6. *Suppose \mathfrak{X} is compact. Let (u_i) be a sequence of solutions of the (perturbed) Fueter equation*

$$\mathfrak{F}u_i = \mathfrak{p} \circ u_i$$

with $\mathfrak{p} \in \Gamma(\mathfrak{X}, \text{Hom}_I(\pi^* TM, V\mathfrak{X}))$ and

$$\mathcal{E}(u_i) := \int_M |\nabla u_i|^2 \leq c_{\mathcal{E}}$$

for some constant $c_{\mathcal{E}} > 0$. Then (after passing to a subsequence) the following holds:

- There exists a closed subset S with $\mathcal{H}^2(S) < \infty$ and a Fueter section $u \in \Gamma(M \setminus S, \mathfrak{X})$ such that $u_i|_{M \setminus S}$ converges to u in C_{loc}^∞ .
- There exist a constant $\varepsilon_0 > 0$ and an upper semi-continuous function $\Theta : S \rightarrow [\varepsilon_0, \infty)$ such that the sequence of measures $\mu_i := |\nabla u_i|^2 \mathcal{H}^4$ converges weakly to $\mu = |\nabla u|^2 \mathcal{H}^4 + \Theta \mathcal{H}^2 \llcorner S$.
- S decomposes as

$$S = \Gamma \cup \text{sing}(u)$$

with

$$\Gamma := \text{supp}(\Theta \mathcal{H}^1 \llcorner S) \quad \text{and} \\ \text{sing}(u) := \left\{ x \in M : \limsup_{r \downarrow 0} \frac{1}{r^2} \int_{B_r(x)} |\nabla u|^2 > 0 \right\}.$$

Γ is \mathcal{H}^2 -rectifiable, and $\text{sing}(u)$ is closed and $\mathcal{H}^2(\text{sing}(u)) = 0$.

- For each smooth point of Γ there exists a non-trivial holomorphic sphere in $\mathfrak{z}_x : S^2 \rightarrow (\mathfrak{X}_x, I(\xi))$ with ξ a unit self-dual 2-form on $T_x M$, whose associated complex structure preserves the splitting $T_x M = T_x \Gamma \oplus N_x \Gamma$. Moreover,

$$\Theta(x) \geq \mathcal{E}(\mathfrak{z}_x) := \int_{S^2} |\mathrm{d}\mathfrak{z}_x|^2.$$

⁷ $\Lambda^+ T^* M$ being trivial is equivalent to $3\sigma(M) + 2\chi(M) = 0$ and $w_2(M) = 0$.

- If \mathfrak{X} is a bundle of simple hyperkähler manifolds with $b_2 \geq 6$, then there is a subbundle $\mathfrak{i} \subset \{I \in \text{End}(TM) : I^2 = -\text{id}\}$, depending only on $\sup \Theta$, whose fibres are finite sets such that $T_x \Gamma$ is complex with respect to a complex structure $I \in \mathfrak{i}_x$ for all smooth points $x \in \Gamma$.

Sketch of the proof. The proof is analogous to that of [Theorem 1.9](#) with a few minor modifications:

- The renormalised energy is now

$$\frac{1}{r^2} \int_{B_r(x)} |\nabla u|^2.$$

- In the proof of the monotonicity formula one now uses the 4-form $\Lambda \in \Omega^4(\mathfrak{X})$ obtained from the section of $\Lambda^+ \pi^* TM \otimes \Lambda^2 V\mathfrak{X}$ induced by I . Direct computation shows that [\(2.7\)](#) still holds. Similarly, one can verify the analogue of [\(2.6\)](#).
- The proof of the ε -regularity and convergence outside S carry over mutatis mutandis.
- In the bubbling analysis, $u_{i;\lambda_i}$ will be asymptotically translation invariant in the direction of $T_x \Gamma$. Fix a unit vector $v_0 \in T_x \Gamma$. Since, asymptotically, everything is invariant in the direction of v_0 , we arrive back at the situation in [Section 7](#). \square

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